Int J Theor Phys (2009) 48: 2987–3000
DOI 10.1007/s10773-009-0094-y

Geodesic Motions in Euclidean Taub-NUT Spinning Spaces

Akhtara Banu · M. Abdullah Ansary

Received: 7 May 2009 / Accepted: 13 July 2009 / Published online: 25 July 2009

Abstract We investigate the geodesic motions of pseudo-classical spin-$\frac{1}{2}$ point particles in the Euclidean Taub-NUT space. We derive the constants of motion from the solutions of the generalized Killing equations for spinning spaces and exploiting those the motion of pseudo-classical Dirac fermions are analyzed on a cone and plane.

Keywords Pseudo-classical Dirac fermions · Symmetries · Killing-Yano tensors · Geodesics

1 Introduction

The relativistic particles endowed with spin-$\frac{1}{2}$ in the geometry of classical metrics are described by pseudo-classical mechanics models involving Grassmann variables ψ^μ for the spin degrees of freedom [1–9]. Although the anti-commuting Grassmann variables do not admit a direct classical interpretation, the Lagrangians for these models have a natural interpretation in the context of the path-integral description of the quantum dynamics. The pseudo-classical equations acquire physical meaning when they are averaged over the inside of the functional integral [2, 10, 11]. In this semiclassical regime, appropriate combinations of Grassmann spin-variables can be replaced by real numbers, neglecting higher order quantum correlations. These ideas have been employed in investigating the motion of spinning point particles in external fields [2, 10, 12–15].

The general relations between spacetime symmetries and the motion of spinning particles have been investigated explicitly in [7–9, 16], which can be exploited to any spacetime. There exists, in addition, a new kind of supersymmetry, which depends on the explicit form

A. Banu (✉)
Department of Mathematics, Rajshahi Govt. College, Rajshahi, Bangladesh
e-mail: akhtara_banu@yahoo.com

A. Banu · M.A. Ansary
Department of Mathematics, Rajshahi University, Rajshahi 6205, Bangladesh

of the metric $g_{\mu\nu}(x)$ [6]. The Euclidean Taub-NUT space admits both kinds of symmetries [17]: *generic* and *nongeneric*.

The complete integrability of particle motion in curved spaces demands the existence of a nontrivial Stäckel-Killing tensor $K_{\mu\nu}$, which gives rise to the associated constant of motion: $Z = \frac{1}{2}K^{\mu\nu}p_\mu p_\nu$, quadratic in the 4-momentum p_μ [18–21]. More surprisingly, the separability of the Klein-Gordon and Dirac equations [22, 23] has the direct consequence of the existence of the Killing-Yano tensor $f_{\mu\nu}$ [24], which is defined as an antisymmetric second-rank tensor satisfying the Penrose-Floyd equation [25, 26]: $D_{(\mu}f_{\nu)\lambda} = 0$, and is a square-root of the Stäckel-Killing tensor: $K^\mu{}_\nu = f^\mu{}_\lambda f^\lambda{}_\nu$. Using supersymmetric particle mechanics Gibbons et al. [6] demonstrated the Killing-Yano tensor as an object belonging to a larger class of possible structures which generate generalized supersymmetry (SUSY) algebras. In the Taub-NUT geometry there exist four Killing-Yano tensors, three of which are complex structures realizing the quaternion algebra. In addition to these three vector-like Killing-Yano tensors, there is a scalar one which has a non-vanishing field strength. The geodesic motion in Taub-NUT space admits a "hidden" symmetry of the Kepler type.

In this paper we investigate the geodesic motion of the pseudo-classical spin-$\frac{1}{2}$ point particle in the geometry of Euclidean Taub-NUT, which is a $D = 4$ self-dual space. The Euclidean Taub-NUT space has attracted much attention in physics. It gives rise to the gravitational analog of the Yang-Mills instanton [27]. The metric of this space is the space part of the line element of the celebrated Kaluza-Klein monopole of Gross and Perry [28] and Sorkin [29]. Moreover, the motion of well-separated monopole-monopole interactions is described approximately by the geodesics of this space [30–33]. The Euclidean Taub-NUT background contains also interesting specific features of the quantum theory in the case of the scalar fields [34] as well as for Dirac fields of spin-$\frac{1}{2}$ fermions [35–37]. There exist large algebras of conserved observables in both cases [38]. The Taub-NUT family of metrics has also attracted physicists in studying many other modern studies like strings, membranes, etc.

The organization of the paper is as follows: We start the next section with a brief account of the relevant equations for the motion of spinning particles in curved spaces. The generalized Killing equations for spinning spaces are analyzed and the constants of motion are derived in terms of the solutions of these equations. In Sect. 3 we investigate the motion of pseudo-classical spinning particles in the NUT-Taub space. We examine the generalized Killing equations for this spinning space and derive the constants of motion in terms of the Killing-Yano tensors. In Sect. 4 we solve the equations derived in the previous section for special case of motion on a cone and on a plane. Finally, in Sect. 5 we present our concluding remarks. We use units: $G = c = 1$.

2 Motion in Spinning Space

Spinning spaces are graded extensions of ordinary Riemannian manifolds; the additional fermionic dimensions are parametrized by vectorial Grassmann coordinates ψ^μ. The geodesic of spinning space is obtained from the action,

$$S = \int_1^2 d\tau \left(\frac{1}{2} g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu + \frac{i}{2} g_{\mu\nu}(x)\psi^\mu \frac{D\psi^\nu}{D\tau} \right), \tag{1}$$

where the covariant derivative of ψ^μ is defined by

$$\frac{D\psi^\mu}{D\tau} = \dot{\psi}^\mu + \dot{x}^\lambda \Gamma^\mu_{\lambda\nu}\psi^\nu, \tag{2}$$

the overdot denotes an ordinary derivative with respect to proper time, $d = d/d\tau$. The equations of motion of the theory can be cast in the form

$$\frac{D^2 x^\mu}{D\tau^2} = \ddot{x}^\mu + \Gamma^\mu_{\lambda\nu}\dot{x}^\lambda \dot{x}^\nu = \frac{1}{2} S^{\kappa\lambda} R_{\kappa\lambda}{}^\mu{}_\nu \dot{x}^\nu, \tag{3}$$

$$\frac{DS^{\mu\nu}}{D\tau} = 0, \tag{4}$$

where the antisymmetric tensor $S^{\mu\nu} = -\mathrm{i}\psi^\mu\psi^\nu$ is the spin-polarization tensor of the particle. The space-like components of the spin-tensor, S^{ij}, are proportional to the particle's magnetic dipole moment, while the time-like components, S^{i0}, represent the electric dipole moment. For free Dirac particles, like free electrons and quarks, the components S^{i0} vanish in the rest frame. This gives the covariant constraint [16]

$$\mathrm{g}_{\nu\lambda}(x) S^{\mu\nu}\dot{x}^\lambda = 0, \tag{5}$$

which takes the form

$$\mathrm{g}_{\mu\nu}(x)\dot{x}^\mu \psi^\nu = 0 \tag{6}$$

in the Grassmann coordinates.

Under the variations

$$\begin{aligned} \delta x^\mu &= \mathcal{R}^\mu(x,\dot{x},\psi) = R^{(1)\mu}(x,\psi) + \sum_{n=1}^{\infty} \frac{1}{n!}\dot{x}^{\nu_1}\cdots\dot{x}^{\nu_n} R^{(n+1)\mu}_{\nu_1\cdots\nu_n}(x,\psi), \\ \delta\psi^\mu &= \mathcal{S}^\mu(x,\dot{x},\psi) = S^{(0)\mu}(x,\psi) + \sum_{n=1}^{\infty} \frac{1}{n!}\dot{x}^{\nu_1}\cdots\dot{x}^{\nu_n} S^{(n)\mu}_{\nu_1\cdots\nu_n}(x,\psi) \end{aligned} \tag{7}$$

the action (1) is left invariant modulo boundary terms, while the Lagrangian is transformed into a total derivative

$$\delta S = \int_1^2 d\tau \frac{d}{d\tau}\left(\delta x^\mu p_\mu - \frac{\mathrm{i}}{2}\delta\psi^\mu \mathrm{g}_{\mu\nu}\psi^\nu - \mathcal{J}(x,\dot{x},\psi)\right), \tag{8}$$

where

$$p_\mu = \Pi_\mu + \frac{\mathrm{i}}{2}\Gamma_{\mu\nu\lambda}\psi^\lambda\psi^\nu, \quad \Pi_\mu = \mathrm{g}_{\mu\nu}\dot{x}^\nu, \tag{9}$$

p_μ is the canonical momentum conjugate to x^μ, and Π_μ the covariant momentum. It follows, from Noether's theorem along with the equations of motion, that the quantity $\mathcal{J}$ is a constant of motion.

The bracket of the world-line Hamiltonian

$$H = \frac{1}{2}\mathrm{g}^{\mu\nu}\Pi_\mu\Pi_\nu \tag{10}$$

with any constant of motion $\mathcal{J}(x,\pi,\psi)$ vanishes:

$$\{H,\mathcal{J}\} = 0 \tag{11}$$

for the Poisson-Dirac brackets defined by

$$\{F,G\} = \mathcal{D}_\mu F\frac{\partial G}{\partial\Pi_\mu} - \frac{\partial F}{\partial\Pi_\mu}\mathcal{D}_\mu G - \mathcal{R}_{\mu\nu}\frac{\partial F}{\partial\Pi_\mu}\frac{\partial G}{\partial\Pi_\nu} + \mathrm{i}(-1)^{a_F}\frac{\partial F}{\partial\psi^\mu}\frac{\partial G}{\partial\psi^\mu}, \tag{12}$$

where

$$\mathcal{D}_\mu = \partial_\mu + \Gamma^\lambda_{\mu\nu}\Pi_\lambda \frac{\partial}{\partial \Pi_\nu} - \Gamma^\lambda_{\mu\nu}\psi^\nu \frac{\partial}{\partial \psi^\lambda}, \qquad \mathcal{R}_{\mu\nu} = \frac{i}{2}\psi^\kappa\psi^\lambda R_{\kappa\lambda\mu\nu}, \tag{13}$$

and a_F is the Grassmann parity of F: $a_F = (0, 1)$ for $F = $ (even, odd). When one uses the expansion

$$\mathcal{J} = \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{J}^{(n)\mu_1\cdots\mu_n}(x,\psi)\Pi_{\mu_1}\cdots\Pi_{\mu_n}, \tag{14}$$

it results then the generalized Killing equations [17]

$$D_{(\mu_{n+1}}\mathcal{J}^{(n)}_{\mu_1\cdots\mu_n)} + \frac{\partial \mathcal{J}^{(n)}_{(\mu_1\cdots\mu_n}}{\partial \psi^\kappa}\Gamma_{\mu_{n+1})\lambda}{}^\kappa \psi^\lambda = \frac{i}{2}\psi^\kappa\psi^\lambda R_{\kappa\lambda\nu(\mu_{n+1}}\mathcal{J}^{(n+1)}_{\mu_1\cdots\mu_n)}{}^\nu, \tag{15}$$

where the parentheses denote full symmetrization with norm one over the indices enclosed. The solutions of these generalized Killing equations are of two classes [6, 7]: *generic* ones, which exist in any spinning particle model (1) and *nongeneric* ones, which depend on the specific background space considered. To the first class belong four independent symmetries given by

(i) Proper-time translations generated by the Hamiltonian H (10);
(ii) SUSY generated by the supercharge

$$Q = \Pi_\mu\psi^\mu; \tag{16}$$

(iii) Chiral symmetry generated by the chiral charge

$$\Gamma_* = \frac{i^{[d/2]}}{d!}\sqrt{g}\,\epsilon_{\mu_1\cdots\mu_d}\psi^{\mu_1}\cdots\psi^{\mu_d}; \tag{17}$$

(iv) Dual SUSY generated by the dual supercharge

$$Q^* = i\{\Gamma_*, Q\} = \frac{i^{[d/2]}}{(d-1)!}\sqrt{g}\,\epsilon_{\mu_1\cdots\mu_d}\Pi^{\mu_1}\psi^{\mu_2}\cdots\psi^{\mu_d} \tag{18}$$

where d is the dimension of the space. In absence of an intrinsic electric dipole moment of physical fermions (leptons and quarks) as formulated in (6), we have

$$Q = 0. \tag{19}$$

To keep the presentation in the following as general as possible, we shall not fix the value of the supercharge.

The *nongeneric* conserved quantities depend on the explicit form of the metric $g_{\mu\nu}(x)$ [6]. The symmetry associated with this type of conserved quantity is generated by the phase-space function

$$Q_f = \mathcal{J}^{(1)\mu}\Pi_\mu + \mathcal{J}^{(0)}, \tag{20}$$

where $\mathcal{J}^{(0,1)}(x,\psi)$ are independent of Π. When (20) is inserted into the generalized Killing equations (15) with $n = 0$, it follows that

$$\mathcal{J}^{(0)}(x,\psi) = \frac{i}{3!}c_{abc}(x)\psi^a\psi^b\psi^c, \tag{21}$$

where

$$c_{abc} = -2D_{[a} f_{bc]}, \qquad D_\mu f_{\nu a} + D_\nu f_{\mu a} = 0, \tag{22}$$

the square brackets denote full antisymmetrization with norm one over the indices enclosed. If there are N such symmetries specified by N sets of tensors $(f_{i\,a}^{\mu}, c_{iabc})$, $i = 1, \ldots, N$, the corresponding generators would be

$$Q_i = f_{i\,a}^{\mu} \Pi_\mu \psi^a + \frac{\mathrm{i}}{3!} c_{iabc}(x) \psi^a \psi^b \psi^c. \tag{23}$$

Obviously, the supercharge (16) is precisely of this form for $f^\mu{}_a = e^\mu{}_a$ and $c_{abc} = 0$. Hence, the choice $i = 0$: $Q = Q_0$, $e^\mu{}_a = f_{0\,a}^{\mu}$, etc., gives the quantities defining the standard SUSY.

The conserved charges Q_i satisfy the algebra

$$\{Q_i, Q_j\} = -2\mathrm{i} Z_{ij}, \tag{24}$$

where

$$Z_{ij} = \frac{1}{2} K_{ij}^{\mu\nu} \Pi_\mu \Pi_\nu + I_{ij}^{\mu} \Pi_\mu + G_{ij}, \tag{25}$$

with $K_{ij\mu\nu}$ the symmetric Stäckel-Killing tensor, $I_{ij\mu}$ the corresponding Killing vector, and G_{ij} the corresponding Killing scalar, respectively, given by

$$K_{ij}^{\mu\nu} = \frac{1}{2} \left(f_{i\,a}^{\mu} f_j^{\nu a} + f_{i\,a}^{\nu} f_j^{\mu a} \right), \tag{26}$$

$$\begin{aligned} I_{ij}^{\mu} &= \frac{1}{2} \mathrm{i} \psi^a \psi^b I_{ijab}^{\mu} \\ &= \frac{1}{2} \mathrm{i} \psi^a \psi^b \left(f_{i\,b}^{\nu} D_\nu f_{j\,a}^{\mu} + f_{j\,b}^{\nu} D_\nu f_{i\,a}^{\mu} + \frac{1}{2} f_i^{\mu c} c_{jabc} + \frac{1}{2} f_j^{\mu c} c_{iabc} \right), \end{aligned} \tag{27}$$

$$\begin{aligned} G_{ij} &= -\frac{1}{4} \psi^a \psi^b \psi^c \psi^d G_{ijabcd} \\ &= -\frac{1}{4} \psi^a \psi^b \psi^c \psi^d \left(R_{\mu\nu ab} f_{i\,c}^{\mu} f_{j\,d}^{\nu} + \frac{1}{2} c_{iab}{}^{e} c_{jcde} \right). \end{aligned} \tag{28}$$

The functions Z_{ij} satisfy the generalized Killing equations (15). Hence their brackets with the Hamiltonian vanish and they are constants of motion, $dZ_{ij}/d\tau = 0$. When $i = j = 0$, (24) reduces to the usual SUSY algebra

$$\{Q, Q\} = -2\mathrm{i} H. \tag{29}$$

If i or j is not equal to zero, Z_{ij} correspond to new bosonic symmetries, unless $K_{ij}^{\mu\nu} = \lambda_{(ij)} g^{\mu\nu}$, with $\lambda_{(ij)}$ a constant (may be zero). The corresponding Killing vector I_{ij}^{μ} and scalar G_{ij} disappear identically. Further, the supercharges for $\lambda_{(ij)} \neq 0$ close on the Hamiltonian. This shows the existence of a second SUSY of the standard type. Thus the theory admits an N-extended SUSY with $N \geq 2$. On the contrary, if there exists a second independent Stäckel-Killing tensor $K^{\mu\nu}$ not proportional to $g^{\mu\nu}$, there exists a genuine new type of SUSY.

The quantity Q_i is a superinvariant

$$\{Q_i, Q\} = 0 \tag{30}$$

for the bracket (12), if and only if

$$K_{0i}^{\mu\nu} = f^{\mu}{}_{a} e^{\nu a} + f^{\nu}{}_{a} e^{\mu a} = 0. \tag{31}$$

In this case, the full constants of motion Z_{ij} can be constructed directly by repeated differentiation of $f^{\mu}{}_{a}$ [6].

Since Z_{ij} are symmetric in (ij) we can diagonalize them. This provides the algebra

$$\{Q_i, Q_j\} = -2\mathrm{i}\delta_{ij} Z_i, \tag{32}$$

where Z_i are $N+1$ conserved bosonic charges of which the first one is the Hamiltonian: $Z_0 = H$.

3 Geodesic Motion in Euclidean Taub-NUT Spinning Spaces

The Euclidean Taub-NUT manifold M_4 is a 4-dimensional Kaluza-Klein space which has static charts with the Cartesian coordinates x^{μ} $(\mu = 1,2,3,4,)$. Here, x^i $(i = 1,2,3)$ are the physical Cartesian space coordinates and x^4 is the Cartesian extra-coordinate. Using the usual three-dimensional vector notations, $\mathbf{x} = (x^1, x^2, x^3)$, $r = |\mathbf{x}|$ and $dl^2 = d\mathbf{x} \cdot d\mathbf{x}$, the line element can be put in the form

$$ds^2 = f(r) dl^2 + \frac{1}{f(r)} [dx^4 + A_i(\mathbf{x}) dx^i]^2, \tag{33}$$

where

$$f(r) = 1 + \frac{\mu}{r}, \qquad A_1 = -\frac{\mu}{r} \frac{x^2}{r + x^3}, \qquad A_2 = \frac{\mu}{r} \frac{x^1}{r + x^3}, \qquad A_3 = 0, \tag{34}$$

μ being a real parameter. If $\mathbf{A}$ is interpreted as the gauge field of a monopole, it results the magnetic field with central symmetry

$$\mathbf{B} = \mu \frac{\mathbf{x}}{r^3}. \tag{35}$$

In the chart of spherical coordinates $(r, \theta, \varphi, \psi)$ where r, θ, φ are commonly related to the physical Cartesian ones x^i, the apparent singularity at the origin is unphysical if x^4 is periodic with period $4\pi\mu$ [39–43]. As a result, the fourth coordinate ψ is defined such that

$$x^4 = -\mu(\psi + \varphi). \tag{36}$$

This chart covers the domain where $r > 0$ for $\mu > 0$ or $r > |\mu|$ for $|\mu| < 0$, the angular coordinates θ, φ cover the sphere S^2 and $0 \le \psi < 4\pi$. Since $A_r = A_\theta = 0$, $A_\varphi = \mu(1 - \cos\theta)$, with $\mu = 2m$ the line element can be put in the form

$$ds^2 = f(r)(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2) + g(r)(d\psi + \cos\theta d\varphi)^2, \tag{37}$$

where

$$f(r) = 1 + \frac{2m}{r}, \qquad g(r) = \frac{4m^2}{f(r)}. \tag{38}$$

The invariance of the metric (37) under spatial rotations and ψ translations is generated by four Killing vectors [39, 41]

$$D^{(\alpha)} \equiv R^{(\alpha)\mu}\partial_\mu, \quad \alpha = 0, \dots, 3,\ \mu = (r, \theta, \varphi, \psi), \tag{39}$$

where

$$\begin{aligned} &R^{(0)} = (0, 0, 0, 1), \qquad R^{(1)} = (0, -\sin\varphi, -\cot\theta\cos\varphi, \csc\theta\cos\varphi), \\ &R^{(2)} = (0, \cos\varphi, -\cot\theta\sin\varphi, \csc\theta\sin\varphi), \qquad R^{(3)} = (0, 0, 1, 0). \end{aligned} \tag{40}$$

$D^{(0)}$, which generates the $U(1)$ of ψ translations, commutes with the other Killing vectors. The remaining three vectors, corresponding to the invariance of the metric (37) under spatial rotations ($\alpha = 1, 2, 3$), obey an $SU(2)$ algebra with

$$[D^{(i)}, D^{(j)}] = -\varepsilon^{ijk} D^{(k)} \quad (i,\ j,\ k = 1,\ 2,\ 3). \tag{41}$$

This is contrasted with the Schwarzschild space, where the isometry group at spacelike infinity is $SO(3) \times U(1)$. This demonstrates the essential topological character of the magnetic monopole mass [44].

These invariances, in the bosonic case, would correspond to the conservation of the so-called “relative electric charge” and the angular momentum [39–42]:

$$q = g(r)(\dot\psi + \cos\theta\dot\varphi), \tag{42}$$

$$\mathbf{j} = \mathbf{r} \times \mathbf{p} + q\frac{\mathbf{r}}{r}, \tag{43}$$

where $\mathbf{p} = f(r)\dot{\mathbf{r}}$ is the “mechanical momentum” canonically conjugate to $\mathbf{r}$.

From the first generalized Killing equation of (15), it follows that to each Killing vector there is associated a Killing scalar. If we limit ourselves to variations (7) that terminate after the terms linear in $\dot x^\mu$, the corresponding constants of motion would be of the form

$$\mathcal{J}^{(\alpha)} = B^{(\alpha)} + m\dot x^\mu R^{(\alpha)}_\mu, \tag{44}$$

which asserts that the Killing scalars $B^{(\alpha)}$ contribute to the “relative electric charge” and the total angular momentum.

For the Euclidean Taub-NUT metric (37), we obtain

$$\begin{aligned} B^{(0)} &= \frac{g'}{2}S^{r\psi} + \frac{g'}{2}\cos\theta S^{r\varphi} - \frac{g}{2}\sin\theta S^{\theta\varphi}, \\ B^{(1)} &= -\frac{g'}{2}\sin\theta\cos\varphi S^{r\psi} - \frac{g}{2}\cos\theta\cos\varphi S^{\theta\psi} + \frac{g}{2}\sin\theta\sin\varphi S^{\varphi\psi} \\ &\quad + \frac{1}{2}(2rf + r^2 f')\sin\varphi S^{r\theta} + (2rf + r^2 f' - g')\sin\theta\cos\theta\cos\varphi S^{r\varphi} \\ &\quad + (g + 2fr^2\cos^2\theta)\cos\varphi S^{\theta\varphi}, \\ B^{(2)} &= -\frac{\partial}{\partial\varphi}B^{(1)}, \\ B^{(3)} &= -g'\cos\theta S^{r\psi} + \frac{g}{2}\sin\theta S^{\theta\psi} - \frac{1}{2}(r^2 f - g)\sin 2\theta S^{\theta\varphi} \\ &\quad + \left[\left(fr + \frac{1}{2}f'r^2\right)\sin^2\theta - g'\cos^2\theta\right]S^{r\varphi}, \end{aligned} \tag{45}$$

where $f' = df/dr$ and $g' = dg/dr$. Then the conserved total angular momentum in the spinning space is given by

$$\boldsymbol{\mathcal{J}} = \mathbf{B} - \mathbf{j}, \qquad \mathcal{J}^{(0)} = B^{(0)} + q \tag{46}$$

with $\boldsymbol{\mathcal{J}} = (\mathcal{J}^{(1)}, \mathcal{J}^{(2)}, \mathcal{J}^{(3)})$ and $\mathbf{B} = (B^{(1)}, B^{(2)}, B^{(3)})$. The components of $\boldsymbol{\mathcal{J}}$ are as follows:

$$\begin{aligned}
\mathcal{J}^{(1)} &= B^{(1)} - r^2 f \sin\varphi\dot{\theta} - r^2 f \cos\theta \sin\theta \cos\varphi\dot{\varphi} - q \sin\theta \cos\varphi, \\
\mathcal{J}^{(2)} &= B^{(2)} + r^2 f \cos\varphi\dot{\theta} - r^2 f \cos\theta \sin\theta \sin\varphi\dot{\varphi} - q \sin\theta \sin\varphi, \\
\mathcal{J}^{(3)} &= B^{(3)} + r^2 f \sin^2\theta\dot{\varphi} - q \cos\theta.
\end{aligned} \tag{47}$$

We obtain, from (48), two interesting relations:

$$\mathcal{J}^{(1)} \sin\varphi - \mathcal{J}^{(2)} \cos\varphi = \frac{1}{2}(2rf + r^2 f')S^{r\theta} + \frac{g}{2} \sin\theta S^{\varphi\psi} - r^2 f\dot{\theta}, \tag{48}$$

$$\begin{aligned}
\mathcal{J}^{(0)} + \frac{\boldsymbol{\mathcal{J}} \cdot \mathbf{r}}{r} &= -\frac{g'}{2} \cos^2\theta S^{r\psi} + \left[3\left(rf + \frac{1}{2}r^2 f'\right) \sin^2\theta - \frac{g'}{2}\right] \cos\theta S^{r\varphi} \\
&\quad + \left[(r^2 f + g) \cos^2\theta + \frac{g}{2}\right] \sin\theta S^{\theta\varphi}.
\end{aligned} \tag{49}$$

In addition to the above constants of motion, there are the four universal conserved charges described in the previous section. In terms of the notation of this section they are given by

(i) The energy

$$E = \frac{1}{2} f\dot{r}^2 + \frac{1}{2} f r^2(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2) + \frac{1}{2g}q^2; \tag{50}$$

(ii) The supercharge

$$Q = f\dot{r}\psi^r + r^2 f(\dot{\theta}\psi^\theta + \sin^2\theta\dot{\varphi}\psi^\varphi) + q(\psi^\psi + \cos\theta\psi^\varphi); \tag{51}$$

(iii) The chiral charge

$$\Gamma_* = r^2 f\sqrt{gf} \sin\theta\psi^r\psi^\theta\psi^\varphi\psi^\psi; \tag{52}$$

(iv) The dual supercharge

$$Q^* = r^2 f\sqrt{gf} \sin\theta(\dot{r}\psi^\theta\psi^\varphi\psi^\psi - \dot{\theta}\psi^r\psi^\varphi\psi^\psi + \dot{\varphi}\psi^r\psi^\theta\psi^\psi - \dot{\psi}\psi^r\psi^\theta\psi^\varphi). \tag{53}$$

The equation of motion formulated in (4) shows that ψ^μ is covariantly constant, from which we obtain

$$\begin{aligned}
\dot{\psi}^\psi &= \left(\frac{g'}{2g}\dot{r} + \frac{g\cot\theta}{2r^2 f}\dot{\theta}\right)\psi^\psi + \left(\frac{2f + rf'}{2rf}\cos\theta\dot{\varphi} - \frac{g'}{2g^2}q\right)\psi^r \\
&\quad + \left[\cos\theta\left(\cot\theta + \frac{1}{2}\tan\theta\right)\dot{\varphi} - \frac{q}{2r^2 f}\cot\theta\right]\psi^\theta \\
&\quad - \cos\theta\left[\left(\frac{g'}{2g} - \frac{2f + rf'}{2rf}\right)\dot{r} + \left(\frac{g\cot\theta}{2r^2 f} - \cot\theta - \frac{1}{2}\tan\theta\right)\dot{\theta}\right]\psi^\varphi,
\end{aligned}$$

$$
\begin{aligned}
\dot{\psi}^r &= -\frac{f'}{2f}\dot{r}\psi^r + \frac{g'}{2gf}q\left(\psi^\psi + \cos\theta\psi^\theta\right) + \frac{r^2 f' + 2rf}{2f}\left(\dot{\theta}\psi^\theta + \sin^2\theta\dot{\varphi}\psi^\varphi\right), \\
\dot{\psi}^\theta &= -\frac{2f + rf'}{2rf}\left(\dot{\theta}\psi^r + \dot{r}\psi^\theta\right) + \sin\theta\cos\theta\dot{\varphi}\psi^\varphi \\
&\quad - \frac{\sin\theta}{2r^2 f}\left[(q + g\cos\theta\dot{\varphi})\psi^\varphi + g\dot{\varphi}\psi^\psi\right], \\
\dot{\psi}^\varphi &= \frac{g}{2r^2 f}\csc\theta\dot{\theta}\psi^\psi - \frac{2f + rf'}{2rf}\dot{\varphi}\psi^r + \left(\frac{q}{2r^2 f}\csc\theta - \cot\theta\dot{\varphi}\right)\psi^\theta \\
&\quad - \left[\frac{2f + rf'}{2rf}\dot{r} + \left(1 - \frac{g}{2r^2 f}\right)\cot\theta\dot{\theta}\right]\psi^\varphi.
\end{aligned}
\tag{54}
$$

The Taub-NUT geometry admits a conserved vector, analogous to the Runge-Lenz vector of the Kepler-type problem [41–43]:

$$
\mathbf{K} = \frac{1}{2}\mathbf{K}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \mathbf{p}\times\mathbf{j} + \left(\frac{q^2}{2m} - 2mE\right)\frac{\mathbf{r}}{r}, \tag{55}
$$

E being the conserved energy given by (50) and $\mathbf{K} = (K^{(1)}, K^{(2)}, K^{(3)})$. The three Stäckel-Killing tensors $K^{(\alpha)}_{\mu\nu}$ $(\alpha = 1, 2, 3)$ are such that $D_{(\lambda}K^{(\alpha)}_{\mu\nu)} = 0$.

Four Killing-Yano tensors are found to exist in the Taub-NUT space, three of which, denoted by f_i $(i = 1, 2, 3)$, are special because they are covariantly constant [17]. In the 2-form notation the explicit expressions for the f_i are given by

$$
f_i = 4m(d\psi + \cos\theta d\varphi)\wedge dx_i - \varepsilon_{ijk}\left(1 + \frac{2m}{r}\right)dx_j\wedge dx_k, \tag{56}
$$

which obey the quaternion algebra

$$
f_i f_j + f_j f_i = -2\delta_{ij}, \qquad f_i f_j - f_j f_i = 2\varepsilon_{ijk} f_k \tag{57}
$$

and the corresponding supercharges are

$$
Q_i = f^{\mu}_{i\,a}\Pi_\mu\psi^a. \tag{58}
$$

The fourth Killing-Yano tensor, which is not trivial and leads to new constants of motion, is given in the 2-form notation by

$$
f_Y = 4m(d\psi + \cos\theta d\varphi)\wedge dr + 4r(r+m)\left(1 + \frac{2m}{r}\right)\sin\theta d\theta\wedge d\varphi. \tag{59}
$$

The field strength contains one independent non-vanishing component, given by

$$
H_{r\theta\varphi} = 2\left(1 + \frac{r}{2m}\right)r\sin\theta, \tag{60}
$$

and the corresponding supercharges have the simple form

$$
Q_Y = f^{\mu}_{Y\,a}\Pi_\mu\psi^a - \frac{i}{3}H_{abc}\psi^a\psi^b\psi^c. \tag{61}
$$

It follows that

$$\{Q_Y, Q_Y\} = -2i\left(H + \frac{\mathcal{J}^2 - \mathcal{J}^{(0)2}}{m^2}\right). \tag{62}$$

In terms of (39), (56) and (59), the components of the Runge-Lenz vector (55) can be written as follows:

$$K_{i\mu\nu} = \frac{1}{2}m\left(f_{Y\mu\lambda}f^{\lambda}_{i\ \nu} + f_{Y\nu\lambda}f^{\lambda}_{i\ \mu}\right) + \frac{1}{4m}\left(R^{(0)}_{\mu}R^{(i)}_{\nu} + R^{(0)}_{\nu}R^{(i)}_{\mu}\right). \tag{63}$$

A detail expression for the components of the Runge-Lenz vector $\mathcal{K}$ in the spinning case is given by

$$\begin{aligned}\mathcal{K}_i = m\Bigg[&\left((f_Y f_i)_{\mu\nu} + \frac{1}{4m^2}R^{(i)}_{(\mu}R^{(0)}_{\nu)}\right)\Pi^{\mu}\Pi^{\nu} + \Bigg(f^{\lambda}_{i\ b}D_{\lambda}f_{Y\mu a} + f^{\lambda}_{i\ \mu}D_{\lambda}f_{Yab}\\ &-\frac{1}{4m^2}(D_b R^{(i)}_{a}R^{(0)}_{\mu} + D_b R^{(0)}_{a}R^{(i)}_{\mu})\Bigg)S^{ab}\Pi^{\mu} + \frac{1}{8m^2}S^{ab}S^{cd}D_b R^{(i)}_{a}D_d R^{(0)}_{c}\Bigg],\end{aligned} \tag{64}$$

which satisfies the following Dirac brackets:

$$\{\mathcal{K}_i, Q_0\} = 0, \qquad \{\mathcal{K}_i, \mathcal{J}_j\} = \varepsilon_{ijk}\mathcal{K}_k, \qquad \{\mathcal{K}_i, \mathcal{K}_j\} = \varepsilon_{ijk}\mathcal{J}_k\left[\frac{\mathcal{J}_0^2}{4m^2} - 2H\right]. \tag{65}$$

The geometrical origin of the nongeneric symmetries generated by the Killing-Yano tensors and the Runge-Lenz vector in the Taub-NUT space is traced and their algebraic structure is described in [17, 45, 46].

4 Special Solutions

In this section we solve the equations derived in the previous section to obtain the full solution of the equations of motion for the usual coordinates x^{μ} and Grassmann coordinates ψ^{μ}. These equations are quite intricate and the general solution is by no means illuminating. Instead of the general solution, we investigate special solutions for the motion on a cone and a plane.

Motion on a Cone

We choose the z-axis along $\mathbf{j}$ so that the motion of the particle may conveniently be described in terms of polar coordinates

$$\mathbf{r} = r\mathbf{e}(\theta, \varphi), \qquad \mathbf{e} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta). \tag{66}$$

For this choice of axis, we have

$$\dot{\theta} = \frac{1}{2r^2 f}(2rf + r^2 f')S^{r\theta} + \frac{g\sin\theta}{2r^2 f}S^{\varphi\psi}, \tag{67}$$

$$\begin{aligned}\dot{\varphi} = &-\frac{q}{r^2 f\cos\theta} - \frac{g'}{2r^2 f\cos\theta}S^{r\psi} - \frac{g}{2r^2 f\sin\theta}S^{\theta\psi}\\ &+\frac{1}{r^2 f}(2rf + r^2 f' - g')S^{r\varphi} + \frac{1}{r^2 f\sin\theta\cos\theta}(2r^2 f\cos^2\theta + g)S^{\theta\varphi}.\end{aligned} \tag{68}$$

In what follows we consider $\dot{\theta} = 0$, which solves $S^{r\theta}$ in terms of $S^{\varphi\psi}$. As a result, it follows that $\Gamma_* = Q^* = 0$. Then, using (55), (67) and (68), the equations of motion for the spin components other than $S^{r\theta}$ or $S^{\varphi\psi}$ are given by

$$\dot{S}^{r\varphi} + \frac{f + rf'}{rf}\dot{r}S^{r\varphi} = -\frac{1}{2}\left[3 + \frac{r^2 g'}{g^2}(2rf + r^2 f')\right]\cot\theta\dot{\varphi}S^{r\theta} + \frac{g'}{2g}r^2\cos\theta\dot{\varphi}S^{\theta\varphi},$$

$$\dot{S}^{\theta\varphi} + \frac{2f + rf'}{rf}\dot{r}S^{r\theta\varphi} = 0,$$

$$\dot{S}^{\theta\psi} - \left(\frac{g'}{2g} - \frac{2f + rf'}{2rf}\right)\dot{r}S^{\theta\psi} = -\frac{1}{2g}\left[3(2rf + r^2 f') + \frac{r^2 fg'}{g}\right]\cos\theta\dot{\varphi}S^{r\theta} - \left(\frac{g'}{2g} - \frac{2f + rf'}{2rf}\right)\cos\theta\dot{r}S^{\theta\varphi}, \tag{69}$$

$$\dot{S}^{r\psi} - \left(\frac{g'}{2g} - \frac{f'}{2f}\right)\dot{r}S^{r\psi} = \frac{1}{2}(3\cot\theta + \tan\theta)\cos\theta\dot{\varphi}S^{r\theta} - \left(\frac{g'}{2g} - \frac{2f + rf'}{2rf}\right)\cos\theta\dot{r}S^{r\varphi} - \frac{r^2 g'}{2g}\cos^2\theta\dot{\varphi}S^{\theta\psi} + \frac{2rf + r^2 f'}{2f}\sin^2\theta\dot{\varphi}S^{\varphi\psi}.$$

A particular solution may be found, if one chooses $S^{r\varphi} = S^{r\psi} = S^{r\theta} = 0$, in the form

$$S^{\theta\varphi} = \frac{C^{\theta\varphi}}{r^2 f}, \qquad S^{\theta\psi} = \frac{\sqrt{g}}{r^2 f}C^{\theta\psi} + \cos\theta\frac{C^{\theta\varphi}}{r^2 f},, \tag{70}$$

where $C^{\theta\varphi}$ and $C^{\theta\psi}$ are Grassmann constants.

The constraint $Q = 0$ (51) yields that $\Gamma_* = Q^* = 0$. For the spin components, one then obtain

$$p_r S^{r\varphi} = qS^{\varphi\psi}, \qquad p_r S^{r\psi} = -p_\varphi S^{\varphi\psi}, \qquad p_r S^{r\theta} = p_\varphi S^{\theta\varphi} + qS^{\theta\psi}, \tag{71}$$

where $\mathbf{p} = f\dot{\mathbf{r}}$. The condition $Q = 0$ modifies drastically the form of the solutions. In spite of the complexity of the equations, we have a simple exact solution for the components of the spin-tensor:

$$S^{\theta\varphi} = \frac{C^{\theta\varphi}}{r^2 f}, \qquad S^{\theta\psi} = \frac{\sqrt{g}}{r^2 f}C^{\theta\psi}. \tag{72}$$

For the equations of motion, we obtain

$$\left[\frac{1}{u^2}\left(\frac{du}{d\varphi}\right)^2 + \sin^2\theta\right]\dot{\varphi}^2 fg = (2gE - q^2)u^2,$$
$$q = \mathcal{J}^{(0)} + \frac{gu^2}{2f}\sin\theta C^{\theta\varphi}, \tag{73}$$
$$\dot{\varphi} = \frac{u^2}{f}\left[-\frac{q}{\cos\theta} - \frac{g\sqrt{g}u^2}{2f\sin\theta}C^{\theta\psi} + \frac{2f\cos^2\theta + gu^2}{f\sin\theta\cos\theta}C^{\theta\varphi}\right],$$

where $u = 1/r$.

Motion on a Plane

As we know, the orbital angular momentum for scalar particles is always conserved, but this is not true for spinning particles. For the latter case only the total angular momentum is a constant of motion. Hence, planar motion for spinning particles happens only in two kinds of situations: (i) the orbital angular momentum vanishes, or (ii) spin and orbital angular momentum are parallel. We consider the plane $\theta = \pi/2$ and discuss the cases separately. From (67) and (68) we obtain

$$S^{r\theta} = -\frac{g}{2rf + r^2 f'} S^{\varphi\psi},$$
$$q = -\frac{g'}{2} S^{r\psi} + g S^{\theta\varphi}. \tag{74}$$

Then the equations of motion for the spin components take the following form:

$$\dot{S}^{r\varphi} + \frac{f + rf'}{rf} \dot{r} S^{r\varphi} = 0,$$
$$\dot{S}^{\theta\varphi} + \frac{2f + rf'}{rf} \dot{r} S^{r\theta\varphi} = 0,$$
$$\dot{S}^{\theta\psi} - \left(\frac{g'}{2g} - \frac{2f + rf'}{2rf}\right) \dot{r} S^{\theta\psi} = 0, \tag{75}$$
$$\dot{S}^{r\psi} - \left(\frac{g'}{2g} - \frac{f'}{2f}\right) \dot{r} S^{r\psi} = \frac{2rf + r^2 f'}{2f} \dot{\varphi} S^{\varphi\psi}.$$

Case (i) The solution describes a particle moving along a fixed radius, for which $\dot{\varphi} = 0$. We obtain the solution

$$S^{r\varphi} = \frac{C^{r\varphi}}{rf}, \qquad S^{\theta\varphi} = \frac{C^{\theta\varphi}}{r^2 f},$$
$$S^{\theta\psi} = \frac{\sqrt{g}}{r^2 f} C^{\theta\psi}, \qquad S^{r\psi} = \frac{\sqrt{g}}{\sqrt{f}} C^{r\psi}. \tag{76}$$

The SUSY constraint $Q = 0$ gives a nenule spin component

$$S^{\theta\varphi} = \frac{C^{\theta\varphi}}{r^2 f}, \tag{77}$$

and consequently, the orbit of the particle is described by

$$\dot{r} = \frac{1}{\sqrt{fg}} \left(2gE - q^2\right)^{\frac{1}{2}},$$
$$q = \mathcal{J}^{(0)} + \frac{g}{2} \frac{C^{\theta\varphi}}{r^2 f}. \tag{78}$$

Case (ii) The concerned motion is for $\dot{\varphi} \neq 0$, and if one chooses $S^{\varphi\psi} = 0$, the solution to the equations of motion for the spin components, (75), is just as given in (76). Interestingly,

$Q=0$ implies even in this case a spin component nenule:

$$S^{\theta\varphi}=\frac{C^{\theta\varphi}}{r^2 f}. \tag{79}$$

Subsequently, for the orbit of the particle, we have

$$\begin{aligned}
\frac{1}{u^2}\left(\frac{du}{d\varphi}\right)^2\dot{\varphi}^2 fg &= (2gE-q^2)u^2,\\
\dot{\varphi} &= \frac{u^2}{f}\mathcal{J}^{(3)}\left(1-\frac{gu^2}{2fq}C^{\theta\varphi}\right)^{-1},\\
q &= \mathcal{J}^{(0)}+\frac{gu^2}{2f}C^{\theta\varphi},
\end{aligned} \tag{80}$$

where $u=1/r$.

5 Concluding Remarks

Our main concern of this study has been the geodesic motion of pseudo-classical Dirac fermions in the four-dimensional Euclidean Taub-NUT space. The supersymmetric extension of the Taub-NUT geometry admits fermionic symmetries along with four standard SUSYs. The appearance of these *nongeneric* SUSYs are closely related to the existence of four Killing-Yano tensors, three of which are complex structures recognizing the quaternion algebra and the Taub-NUT manifold is hyper-Kähler [41]. Beside these three vector-like Killing-Yano tensors, there is a scalar one which has a nonvanishing field strength and which exists by virtue of the metric being type D. With a plentiful symmetries the family of Taub-NUT metrics provides an excellent background to analyze the classical and quantum conserved quantities on curved spaces.

We have described the conserved quantities of the Euclidean Taub-NUT spinning-space spanned by $\{x^\mu,\psi^\mu\}$ and obtained the geodesic equations for the motion of pseudo-classical spin one half particles with the spin characterized by the anticommuting spin-polarization tensor $S^{\mu\nu}=-i\psi^\mu\psi^\nu$, ψ^μ being anticommuting Grassmann coordinates. The conserved quantities admit contributions from the spin variables. In spite of the complexity of the equations, we are able to present special solutions for the motion on a cone and on a plane. The supersymmetric constraint $Q=0$ (51) plays an important role for the forms of solutions.

The results show spin dependence of the orbits of the particles in a gravitational field. This leads to the existence of a gravitational analogue of the Stern-Gerlach-type forces well known to appear in electromagnetic phenomena.

Generalizations of Riemannian geometry based on anticommuting variables have been proved to be of wide mathematical interest; for example, supersymmetric point particle mechanics has found applications in the area of index theorem, while the BRST methods are widely used in the study of topological invariants. It is, therefore, well motivated to study the geometry of graded pseudo-manifold with the coordinates $\{x^\mu,\psi^\mu\}$.

References

1. Barducci, A., Casalbuoni, R., Lusanna, L.: Nuovo Cimento A **35**, 377 (1976)
2. Berezin, F.A., Marinov, M.S.: Ann. Phys. (NY) **104**, 336 (1977)

3. Brink, L., Deser, S., Zumino, B., Vecchia, P.Di., Howe, P.: Phys. Lett. B **64**, 435 (1976)
4. Brink, L., Vecchia, P.Di., Howe, P.: Nucl. Phys. B **118**, 76 (1977)
5. Casalbuoni, R.: Phys. Lett. B **62**, 49 (1976)
6. Gibbons, G.W., Rietdijk, R.H., van Holten, J.W.: Nucl. Phys. B **404**, 42 (1993)
7. Rietdijk, R.H., van Holten, J.W.: Class. Quantum Gravity **7**, 247 (1990)
8. Rietdijk, R.H., van Holten, J.W.: Class. Quantum Gravity **10**, 575 (1993)
9. van Holten, J.W., Rietdijk, R.H.: J. Geom. Phys. **11**, 559 (1993)
10. Berezin, F.A., Marinov, M.S.: Pis'ma Zh. Eksp. Teor. Fiz. **21**, 678 (1975); English transl. JETP Lett. **21**, 320 (1975)
11. Barducci, A., Casalbuoni, R., Lusanna, L.: Nucl. Phys. B **180**(FS2), 141 (1981)
12. Barducci, A., Bondi, F., Casalbuoni, R.: Nuovo Cimento B **64**, 287 (1981)
13. Barducci, A., Casalbuoni, R., Lusanna, L.: Nucl. Phys. B **124**(93), 521 (1977)
14. van Holten, J.W.: Nucl. Phys. B **356**, 3 (1991)
15. van Holten, J.W.: Phys. A **182**, 279 (1992)
16. Rietdijk, R.H.: Applications of supersymmetric quantum mechanics. Ph.D. thesis, Amsterdam (1992)
17. van Holten, J.W.: Phys. Lett. B **342**, 47 (1995)
18. Carter, B.: Commun. Math. Phys. **10**, 280 (1968)
19. Carter, B.: Phys. Rev. **174**, 1559 (1968)
20. Carter, B.: Phys. Rev. D **16**, 3395 (1977)
21. Walker, M., Penrose, R.: Commun. Math. Phys. **18**, 265 (1970)
22. Chandrasekhar, S.: Proc. R. Soc. A **349**, 571 (1976)
23. Carter, B., McLenaghan, R.G.: Phys. Rev. D **19**, 1093 (1979)
24. Yano, K.: Ann. Math. **55**, 328 (1952)
25. Penrose, R.: Ann. NY Acad. Sci. **224**, 125 (1973)
26. Floyd, R.: The dynamics of Kerr fields. Ph.D. thesis, London (1973)
27. Hawking, S.W.: Phys. Lett. A **60**, 81 (1977)
28. Gross, D.J., Perry, M.J.: Nucl. Phys. B **226**, 29 (1983)
29. Sorkin, R.D.: Phys. Rev. Lett. **51**, 87 (1983)
30. Manton, N.S.: Phys. Lett. B **110**, 54 (1982)
31. Manton, N.S.: Phys. Lett. B **154**, 397 (1985); Erratum, Phys. Lett. B **157**, 475 (1985)
32. Atiyah, M.F., Hitchin, N.: Phys. Lett. A **107**, 21 (1985)
33. Atiyah, M.F., Hitchin, N.J.: The Geometry and Dynamics of Magnetic Monopoles. Princeton University Press, Princeton (1987)
34. Cotăescu, I.I., Visinescu, M.: Mod. Phys. Lett. A **15**, 145 (2000)
35. Cotăescu, I.I., Visinescu, M.: Int. J. Mod. Phys. A **16**, 1743 (2001)
36. Cotăescu, I.I., Visinescu, M.: Phys. Lett. B **502**, 229 (2001)
37. Cotăescu, I.I., Visinescu, M.: Class. Quantum Gravity **18**, 3383 (2001)
38. Cotăescu, I.I., Visinescu, M.: J. Math. Phys. **43**, 2987 (2002)
39. Gibbons, G.W., Manton, N.S.: Nucl. Phys. B **274**, 183 (1986)
40. Gibbons, G.W., Ruback, P.J.: Phys. Lett. B **188**, 226 (1987)
41. Gibbons, G.W., Ruback, P.J.: Commun. Math. Phys. **115**, 267 (1988)
42. Feher, L.Gy., Horvathy, P.A.: Phys. Lett. B **183**, 182 (1987); Erratum, Phys. Lett. B **188**, 512 (1987)
43. Cordani, B., Feher, L.Gy., Horvathy, P.A.: Phys. Lett. B **201**, 481 (1988)
44. Mueller, M., Perry, M.J.: Class. Quantum Gravity **3**, 65 (1986)
45. Vaman, D., Visinescu, M.: Phys. Rev. D **54**, 1398 (1996)
46. Vaman, D., Visinescu, M.: Phys. Rev. D **57**, 3790 (1998)